A multiobjective interval portfolio framework for supporting investor's preferences under different risk assumptions

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Abstract

This paper is aimed at presenting a multiobjective portfolio framework which considers the intrinsic uncertainty of investment decisions under different risk assumptions (i.e. mean-absolute deviation and mean-semi-absolute deviation), where the expected return and risk of the assets are treated as interval numbers. Some realistic interval constraints such the maximal and minimal fractions of the capital allocated to the various assets are also considered. New surrogate problems are obtained for the mean-absolute deviation risk measure based on the concept of necessary subtraction between interval numbers. A proposal for obtaining the efficient portfolio solutions is also suggested, which allows considering three types of investment strategies, i.e., a conservative strategy, an aggressive strategy and a combined strategy. A sample of ten years (about 3600 trading days) of data regarding a diversified portfolio of stocks has been collected which allows illustrating the applicability of the approach proposed. Results illustrate the trade-off between risk and return, being also consistent with the type of strategy followed by the investor, i.e. more aggressive strategies toward risk lead to lower risk levels and more aggressive strategies toward return lead to higher return and vice-versa. Additionally, we conclude that less prone to risk investors might find the formulation based on the mean-absolute necessary deviation more attractive since it allows reaching the highest return values in the worst case scenarios. Finally, results indicate that if a conservative strategy is followed the portfolios obtained are always more diversified.

Keywords: Portfolio theory, Multiobjective interval linear portfolio problems, mean-absolute deviation-risk models

1. Introduction

In 1952, Markowitz (1952) paved the grounds for modern portfolio theory, with the application of variance or standard deviation as a measure of risk.

The classical Markowitz formulation is valid as long as the expected return is multivariate normally distributed and the investor is risk averter and prefers lower risk (Papahristodoulou and Dotzauer, 2004). This approach has drawn many criticisms, since it might allow for the choice of a portfolio that might be outperformed and because of its complexity (the objective function is quadratic), being very hard to find an optimal portfolio when the number of assets is large. Additionally, even if an optimal solution is obtained, it might be hard to implement it in practice, eventually leading the investor to assign his/her budget to a large number of small blocks of shares, which might be unprofitable, given the transaction costs. Finally, the standard Markowitz' formulation, considers variance as a measure of risk volatility, disregarding the fact that quite often the investor's stance regarding risk is not consistent with symmetry or normal distribution. A small loss might be enough to make an investor not prone to risk very unhappy and the opposite might also be true. The semi-variance was also proposed by Markowitz (1968), providing a better risk assessment than the measure originally considered, but involving a higher computational burden and disclosing the same type of information.

The Markowitz classical model should thus be considered as an approximation to rather complex problems that investors have to face. Real-world portfolios are made up of a large number of assets, possibly with very small holdings for some of them, minimum lot sizes, complexity of management, or policy of the asset management companies (Cesarone et al., 2011).

The classical Markowitz model has evolved and some variations therefrom have been comprehensively studied in the past decade, particularly from the computational standpoint. Several attempts have been made to build less complex portfolio selection problems by linearizing the quadratic objective function (see e.g. Sharpe (1971); Speranza (1993); Mansini et al. (2003)). Usually the approximation or the decomposition of the covariance matrix are considered (Mitra et al. (2003)). Alternative risk measures for portfolio planning have also been used (for an overview on distinct risk estimation measures usually accounted for in the framework of portfolio theory see Biglova et al. (2004); Ortobelli et al. (2005)).

Although a prevalent number of publications exists regarding risk measures and mean-risk models, portfolio decisions may also be based on the investor's expectations regarding return, risk and liquidity characteristics of the assets (Gupta et al., 2014). Investors may be interested in grasping how different assets may be combined in order to obtain the aimed return, risk and liquidity.

Conventional multiobjective models usually address practical portfolio selection problems in which all coefficients and parameters are *a priori* given. Nevertheless, in real-world portfolio problems, information regarding the asset returns, risk and liquidity is often incomplete, the markets in which the assets are traded exhibit volatility and experts' opinions might vary. Therefore, besides multiple issues of evaluation, these problems inherently involve inexactness and uncertainty issues. Uncertainty handling can be dealt with in various ways, namely by means of stochastic, fuzzy and interval programming techniques. In the stochastic approach the coefficients are treated as random variables with known probability distributions. In the fuzzy approach, the constraints and objective functions are regarded as fuzzy sets with known membership functions. However, it is not always easy for the decision-maker (DM) to specify these probability distributions and membership functions. Thus it would be more realistic to define portfolio parameters in terms of intervals rather than crisp numbers. In the interval approach it is considered that the uncertain values are perturbed simultaneously and independently within known fixed bounds, being therefore intuitively preferred by the DM in practice (Oliveira and Antunes, 2007). In this framework, several authors propose portfolio selection models based on the traditional semi-absolute deviation measure of risk, by taking the uncertain returns of assets in a financial market as intervals (see e.g. Lai et al. (2002); Bhattacharyya et al. (2011); Zhang (2016)). The minimax regret approach based on a regret function has also been considered in a portfolio selection problem in which the prices of the securities are treated as interval variables (Giove et al., 2006). Multi-period portfolio selection models with interval coefficients have also been suggested (see e.g. Liu et al. (2013); Zhang (2016); Liu et al. (2016)). Other combined approaches have also been proposed which present a new portfolio selection model, where the average return of every asset is considered as an interval number, and the risk of every asset is treated by a probabilistic measure (Jin, 2016)).

Usually the portfolio selection models herein reviewed were applied assuming that an investor decides to invest his/her wealth among a small number of assets, considering the explicit uncertainty of the model mainly in its objective functions. Therefore, we suggest a methodological approach to assist investors regarding portfolio decisions, which allows reducing the complexity of the problem from a computational standpoint even when the consideration of uncertainty involves both the objective functions and the constraints of the model.

Two multiobjective portfolio optimization models with interval coefficients are proposed with different risk assumptions, where interval constraints regarding the maximal fractions of the capital allocated to the various assets will also be considered to guarantee portfolio diversification. New surrogate problems are then obtained for the mean-absolute deviation risk measure based on the concept of necessary subtraction between interval numbers. Finally, in order to obtain efficient portfolio solutions, three types of investment strategies will be explored, i.e., a conservative strategy, an aggressive strategy and a combined strategy.

The remaining of this paper is organized as follows. In Section 2 we briefly describe the methodological approach used. Section 3 describes the main underpinning assumptions regarding data collection. In Section 4 a discussion of the illustrative results obtained is presented. Finally, some conclusions are drawn and future work developments are suggested.

2. Methodology and assumptions

Let us assume that investors allocate their wealth among n assets offering random rate of returns and that the portfolio selection problem considered is based on a single period model of investment.

In the next sections the assumptions used in the construction of the models suggested are presented, the objective functions and the constraints considered are also described which allow obtaining two different modelling approaches according to different risk measures. Finally, three different mathematical models are obtained for each of these models according to different investor's standpoints.

2.1. Objective functions

A portfolio is composed of two or more assets represented by an ordered n-tuple $\Theta = (x_1, x_2, \ldots, x_n)$, where x_i is the proportion of the total funds invested in the i-th asset.

Return

The asset's return or the rate of return is defined for a given period t as:

$$r_{it} = \frac{(p_{it})-(p_{it-1})+(d_{it})}{(p_{it-1})} \tag{1}$$

where p_{it} is the closing price of the i-th asset during the period t, p_{it-1} is the closing price during the period $t-1$, d_{it} is the dividend of the i-th asset during the period t.

The expected value ($E[\cdot]$) of the rate of return, R_i (i = 1, 2, . . . , n), is a random variable taking finitely many values can also be approximated by the average derived from historical data, i.e.,

$$r_i = E[R_i] = \frac{1}{T}\sum_{t=1}^{T} r_{it}. \tag{2}$$

In general, the arithmetic mean of past returns is considered as a proxy of expected return of an asset and thus it is obtained as a certain value. However, in real world problems, asset prices and the returns obtained therefrom are subject to a set of variables whose behavior cannot be simply anticipated on past events (Gupta et al., 2014). Moreover, the use of arithmetic mean of historical returns as the expected return, has two major shortcomings. On one hand, if historical data for a long period of time are considered, the influence of the earlier historical data is the same as that of recent past data. Nevertheless, recent past data of an asset might be more significant than the earlier historical data. On the other hand, if the historical data of an asset are not suitable, due to lack of information, the estimation of the statistical parameters would not be adequate. Having this in mind, in order to account for the uncertainty handling, the expected return of an asset should be rather considered as an interval number.

Therefore, the return of the portfolio is expressed as:

$$\sum_{i=1}^{n}[r_i^L, r_i^U]x_i = [\sum_{i=1}^{n} r_i^L x_i, \sum_{i=1}^{n} r_i^U x_i], \tag{3}$$

where $[r_i^L, r_i^U]$ is the interval valued return with.

Risk

Usually, an investor would rather prefer to have the portfolio return as large as possible and at the same time with minimum possible dispersion/variability. Therefore, Markowitz (1952) suggested the variance to quantify portfolio risk.

Although, variance can be used as a risk measure, one of its main limitations is that it penalizes extreme upside (gains) and downside (losses) deviations from the expected return. Thus, when probability distributions of the asset returns are asymmetric, variance becomes a less appropriate measure of portfolio risk (Chunhachinda, et al. 1997). In fact, the selected portfolio may sacrifice higher expected returns.

Markowitz (1968) also suggested the semi-variance which is a downside risk measure, i.e., a measure which only considers the negative deviations from a reference return level. Its advantage over variance is that it does not consider gains as risk; thus, it is a suitable measure of risk when investors are concerned about portfolio underperformance rather than over performance. Nevertheless, the implementation of mean-semi-variance portfolio selection models is computationally much more complex as compared to mean-variance portfolio selection models.

In 1994 JP Morgan suggested another risk measure also known as Value-at-Risk (see Longerstaey and Spencer (1996)), which entails several drawbacks since the Value-at-Risk optimization problem is not convex and it does not allow expressing the benefits of diversification (Cesarone et al., 2011). Other important risk measure is Conditional Value-at-Risk also called Mean Excess Loss, Mean Shortfall, or Tail VaR (Rockafellar and Uryasev, 2000).

A surrogate measure for risk is the maximization of the minimum return (maximum loss) demanded by the investor (Young, 1998):

$$\text{Max min} \sum_{i=1}^{n} r_{it} x_i, \ t = 1, ..., T. \tag{4}$$

Young (1998) argued that for given distributions, in particular when data is log-normally distributed, or skewed, this type of formulation might be rather preferable. This author also advocated its use when the portfolio optimization problem involves a large number of decision variables (including integer variables), or if the investor’s is more risk averse than it is implied by the classical minimization of variance. However, in spite of its simplicity this formulation might lead to an infeasible solution if all assets yield a negative return.

A different approach to replace the Markowitz classic formulation is to use the absolute deviation risk function (Konno and Yamazaki, 1991; Mansini and Speranza, 1999; Rudolf et al. 1999). The absolute deviation of a random variable is the expected absolute value of the difference between the random variable and its mean. The portfolio risk measured as absolute deviation can be approximated as follows:

$$\frac{1}{T}\sum_{t=1}^{T}\left|\sum_{i=1}^{n}(r_{it}-r_i)x_i\right| \quad . \tag{5}$$

Since the expected returns of assets are considered as interval numbers, the expected absolute deviation of return of the portfolio below the expected return is an interval number too:

$$\frac{1}{T}\sum_{t=1}^{T}\left|\sum_{i=1}^{n}\left(r_{it}-[r_i^L,r_i^U]\right)x_i\right| \tag{6}$$

Konno and Yamazaki (1991) concluded that if the return is multivariate normally distributed, the minimization of the absolute deviation provides similar results to the classical Markowitz formulation. Rudolf et al. (1999) argue that the minimization of the absolute deviation is equivalent to expected utility maximization under risk aversion. This formulation has several advantages since it does not require the estimation of the variance-covariance matrix and the solution is obtainable even if all possible assets yield a negative return. This model is also flexible enough to be reformulated as an Integer Linear Programming (ILP) problem incorporating other important features (e.g. fixed and variable costs associated with the purchase of assets) and decision variables (Mansini and Speranza, 1999), being easily implemented even when a large number of assets is considered.

Following the work of Konno and Yamazaki (1991), Speranza (1993) proposed the semi-absolute deviation as an alternative measure to quantify risk, and concluded that considering the risk function as a linear combination of the mean-semi-absolute deviations (i.e., mean deviations below and above the portfolio return), a model equivalent to the mean-absolute deviation model can be obtained, whenever the sum of the coefficients of the linear combination is positive. Finally, they have also showed that this model is equivalent to the Markowitz model, if the returns are normally distributed.

The expected semi-absolute deviation of return of the portfolio below the expected return is given by:

$$\sum_{t=1}^{T} \frac{\left|\sum_{i=1}^{n}(r_{it}-r_i)x_i\right|+\sum_{i=1}^{n}(r_i-r_{it})x_i}{2T} \tag{7}$$

The mean-semi-absolute deviation model reduces the number of constraints by half in comparison with the mean-absolute deviation model, since it requires T linearizing constraints while the mean-absolute deviation model requires 2T linearizing constraints.

If the expected returns of assets are given as interval numbers, the expected semi-absolute deviation of return of the portfolio below the expected return is an interval number too:

$$\left[\sum_{t=1}^{T} \frac{\left|\sum_{i=1}^{n}(r_{it}-r_i^L)x_i\right|+\sum_{i=1}^{n}(r_i^L-r_{it})x_i}{2T}, \sum_{t=1}^{T} \frac{\left|\sum_{i=1}^{n}(r_{it}-r_i^U)x_i\right|+\sum_{i=1}^{n}(r_i^U-r_{it})x_i}{2T}\right] \tag{8}$$

2.2. Constraints

Capital budget constraint

Since x_i is the proportion of the total funds invested, the capital budget constraint on the assets is expressed as:

$$\sum_{i=1}^{n} x_i=1. \tag{9}$$

Maximum proportion of capital that can be invested

The maximum proportion of capital allocated to the assets in the portfolio depend upon several factors. Since investors differ in their interpretation of the available information, in order to achieve a sufficient diversification of investments we consider the upper bound given within interval, $[u_i^L, u_i^U]$, obtaining the following interval constraint:

$$\sum_{i=1}^{n} x_i \leq [u_i^L, u_i^U]y_i, \quad i = 1, \ldots, n, \tag{10}$$

where y_i is a binary variable indicating whether the i-th asset is contained in the portfolio.

Maximum number of assets held in a portfolio

Investors may differ regarding the number of assets they want to manage in a portfolio. Therefore, we consider that the maximum number of assets given within an interval range, $[h^L, h^U]$, and obtain the following constraint:

$$\sum_{i=1}^{n} y_i \leq [h^L, h^U]. \quad (11)$$

No short selling is allowed

Short selling occurs when an investor actually does not own an asset but he/she establishes a market position by selling the asset in anticipation that the price of that asset will fall. In this case, the investor is said to have taken a short position. Mathematically, this situation implies that the number of assets owned by the investor is negative.

In portfolio mathematical modelling, short selling is not allowed, i.e. the values of x_i are not negative; hence,

$$x_i \geq 0 \text{ for all i (i = 1, 2, . . . , n).} \quad (12)$$

2.3. Multiobjective Portfolio Selection Problems Using Interval Numbers

Two interval portfolio optimization problems are obtained considering different risk assumptions.

Mean-absolute deviation model

$$\text{Min } \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{i=1}^{n}\left(r_{it} - [r_i^L, r_i^U]\right)x_i\right|$$

$$\text{Max } \left[\sum_{i=1}^{n} \mathrm{r_i^L}\, x_i, \sum_{i=1}^{n} \mathrm{r_i^U}\, x_i\right],$$

$$\text{s.t. } \sum_{i=1}^{n} x_i = 1,$$

$$\sum_{i=1}^{n} y_i \leq [h^L, h^U],$$

$$x_i \leq [u_i^L, u_i^U] y_i, \text{ i = 1, …, n,}$$

$$x_i \geq 0, \text{ i = 1, …, n,}$$

$$y_i \in \{0,1\}, \text{ i = 1, …, n.} \quad (13)$$

If we consider the necessary subtraction operator (for the properties of necessary subtraction see Inuiguchi and Kume (1991)), the absolute necessary deviation of the return obtained from the expected interval returns in each period is:

$$E_t(\mathbf{x}) = \begin{cases} \left\| \left[\sum_{i=1}^{n} r_i^L x_i - \sum_{i=1}^{n} r_{it}\, x_i, \sum_{i=1}^{n} r_i^U x_i - \sum_{i=1}^{n} r_{it}\, x_i \right] \right\| & if\ \sum_{i=1}^{n} r_i^U x_i - \sum_{i=1}^{n} r_i^L x_i \geq 0, \\ \left\| \left[\sum_{i=1}^{n} r_{it}\, x_i - \sum_{i=1}^{n} r_i^L x_i\, , \sum_{i=1}^{n} r_{it}\, x_i - \sum_{i=1}^{n} r_i^U x_i \right] \right\| & if\ \sum_{i=1}^{n} r_i^U x_i - \sum_{i=1}^{n} r_i^L x_i \leq 0. \end{cases} =$$

$$= \begin{cases} \left[\sum_{i=1}^{n} r_i^L x_i - \sum_{i=1}^{n} r_{it}\, x_i\, , \sum_{i=1}^{n} r_i^U x_i - \sum_{i=1}^{n} r_{it}\, x_i\right] \ if\ \sum_{i=1}^{n} r_i^U x_i \geq \sum_{i=1}^{n} r_i^L x_i \geq 0, \\ \left[0, \left(\sum_{i=1}^{n} r_i^U x_i - \sum_{i=1}^{n} r_{it}\, x_i\right) \vee \left(\sum_{i=1}^{n} r_{it}\, x_i - \sum_{i=1}^{n} r_i^L x_i\right)\right] \ if\ \sum_{i=1}^{n} r_i^L x_i < 0 < \sum_{i=1}^{n} r_i^U x_i\, , \\ \left[\sum_{i=1}^{n} r_{it}\, x_i - \sum_{i=1}^{n} r_i^U x_i\, , \sum_{i=1}^{n} r_{it}\, x_i - \sum_{i=1}^{n} r_i^L x_i\right] \ if\ \sum_{i=1}^{n} r_i^L x_i \leq \sum_{i=1}^{n} r_i^U x_i \leq 0. \end{cases} \quad (14)$$

since $\sum_{i=1}^{n} r_i^U x_i - \sum_{i=1}^{n} r_i^L x_i \geq\ 0$ is allways verified.

The necessary regret interval can be obtained as:

$$E_t(\mathbf{x}) = [(e_t^{L-} +\ e_t^{U+}) \wedge (\, e_t^{L+} + e_t^{U-}), (e_t^{L-} \vee e_t^{U-} \vee e_t^{L+} \vee\, e_t^{U+})] \quad (15)$$

where the deviational variables, e_t^{L-}, e_t^{L+} , e_t^{U+} and e_t^{U-} are defined in such a way that:

$$\sum_{i=1}^{n} r_{it}\, x_i + e_t^{L-} - e_t^{L+} = \sum_{i=1}^{n} r_i^L x_i, \quad (16)$$

$$\sum_{i=1}^{n} r_{it}\, x_i + e_t^{U-} - e_t^{U+} = \sum_{i=1}^{n} r_i^U x_i, \quad (17)$$

where $e_t^{U-} e_t^{U+} = 0$, $e_t^{L-} e_t^{L+} = 0$ and the "$\wedge$" and "$\vee$" operators are the minimum and maximum values, respectively.

By considering the upper bound of the necessary deviation of the return of each portfolio from its corresponding interval expected returns, the smallest deviation is guaranteed (Inuiguchi and Kume, 1991; Oliveira and Antunes, 2017) and problem (13) has the following surrogate problem:

Min $\lambda \frac{1}{T} \sum_{t=1}^{T} v_t + (1-\lambda) u^U$,

Max $[\sum_{i=1}^{n} r_i^L\, x_i, \sum_{i=1}^{n} r_i^U\, x_i]$,

s.t. $\sum_{i=1}^{n} x_i$=1,

$\sum_{i=1}^{n} y_i \leq [h^L\, , h^U]$,

$x_i \leq [u_i^L, u_i^U] y_i$, i = 1, …, n,

$\sum_{i=1}^{n} r_{it}\, x_i + e_t^{L-} - e_t^{L+} = \sum_{i=1}^{n} r_i^L x_i$, t=1, ..., T,

$\sum_{i=1}^{n} r_{it}\, x_i + e_t^{U-} - e_t^{U+} = \sum_{i=1}^{n} r_i^U x_i$, t=1, ..., T,

$e_t^{U-} + e_t^{U+} \leq v_t$, t=1, 2, …, T,

$e_t^{L-} + e_t^{L+} \leq v_t$, t=1, 2, …, T,

$v_t \leq u^U$,

$x_i \geq 0$, i = 1, …, n,

$y_i \in \{0,1\}$, i = 1, …, n,

$0 \leq \lambda \leq 1$,

$v_t \geq 0,\ \mathrm{t} = 1, 2, \dots, \mathrm{T}$. (18)

Mean-semi-absolute deviation model

Min $[\sum_{t=1}^{T} \frac{|\sum_{i=1}^{n}(r_{it}-r_i^L)x_i| + \sum_{i=1}^{n}(r_i^L - r_{it})x_i}{2T}, \sum_{t=1}^{T} \frac{|\sum_{i=1}^{n}(r_{it}-r_i^U)x_i| + \sum_{i=1}^{n}(r_i^U - r_{it})x_i}{2T}]$,

Max $\left[\sum_{i=1}^{n} \mathrm{r}_\mathrm{i}^\mathrm{L}\, x_i, \sum_{i=1}^{n} \mathrm{r}_\mathrm{i}^\mathrm{U}\, x_i\right]$,

s.t. $\sum_{i=1}^{n} x_i$=1,

$\sum_{i=1}^{n} y_i \leq \left[h^L, h^U\right]$,

$x_i \leq [u_i^L, u_i^U] y_i$, i = 1, …, n,

$x_i \geq 0$, i = 1, …, n,

$y_i \in \{0,1\}$, i = 1, …, n. (19)

Problem (19) has the following surrogate:

Min $[\frac{1}{T}\sum_{t=1}^{T} p_t^1, \frac{1}{T}\sum_{t=1}^{T} p_t^2]$,

Max $[\sum_{i=1}^{n} r_i^L\, x_i, \sum_{i=1}^{n} r_i^U\, x_i]$,

s.t. $\sum_{i=1}^{n} x_i$=1,

$\sum_{i=1}^{n} y_i \leq \left[h^L, h^U\right]$,

$x_i \leq [u_i^L, u_i^U] y_i$, i = 1, …, n,

$p_t^1 + \sum_{i=1}^{n}(r_{it} - r_i^L)\, x_i \geq 0$, t=1, 2, …, T,

$p_2^1 + \sum_{i=1}^{n}\left(r_{it} - r_i^U\right) x_i \geq 0$, t=1, 2, …, T,

$x_i \geq 0$, i = 1, …, n,

$y_i \in \{0,1\}$, i = 1, …, n,

$p_t^1 \geq 0$, t=1, 2, …, T,

$p_t^2 \geq 0$, t=1, 2, …, T. (20)

2.4. Solution method

Problems (18) and (20) can be transformed into multiobjective mixed integer interval linear programming problems. The weighted-sum method can thus be used to convert the multiobjective problems into single interval objective optimization problems (see e.g. Gupta et al. (2014)). We suggest distinct optimization models for portfolio selection regarding three types of investment strategies: conservative, aggressive and combined strategies.

2.4.1. Conservative strategy

The investor aiming for a conservative strategy is more risk averse, being more concerned with risk than return.

Mean-absolute deviation model (with a pessimistic stance regarding risk)

Max $(\beta(\sum_{i=1}^{n} r_i^L x_i)$ - α($\lambda \frac{1}{T}\sum_{t=1}^{T} v_t + (1-\lambda)u^U$))

s.t. $\sum_{i=1}^{n} x_i$=1,

$\sum_{i=1}^{n} y_i \leq h^L$,

$x_i \leq u_i^L y_i$, i = 1, …, n,

$\sum_{i=1}^{n} r_{it}\, x_i + e_t^{L-} - e_t^{L+}$=$\sum_{i=1}^{n} r_i^L x_i$, t=1, ..., T,

$\sum_{i=1}^{n} r_{it}\, x_i + e_t^{U-} - e_t^{U+}$=$\sum_{i=1}^{n} r_i^U x_i$, t=1, ..., T,

$e_t^{U-} + e_t^{U+} \leq v_t$, t=1, 2, …, T,

$e_t^{L-} + e_t^{L+} \leq v_t$, t=1, 2, …, T,

$v_t \leq u^U$,

$x_i \geq 0$, i = 1, …, n,

$y_i \in \{0,1\}$, i = 1, …, n,

$0 \leq \lambda \leq 1$,

$v_t \geq 0$ t = 1, 2, ..., T. (21)

Mean-semi-absolute deviation model (with a pessimistic stance regarding risk)

Max $(\varphi\sum_{i=1}^{n} r_i^L x_i - \tau\, \frac{1}{T}\sum_{t=1}^{T} p_t^2)$, s.t. $\sum_{i=1}^{n} x_i$=1,

$\sum_{i=1}^{n} y_i \leq h^L$,

$x_i \leq u_i^L y_i$, i = 1, …, n,

$p_2^1 + \sum_{i=1}^{n}\left(r_{it} - r_i^U\right) x_i \geq 0$, t=1, 2, …, T,

$x_i \geq 0$, i = 1, …, n,

$y_i \in \{0,1\}$, i = 1, …, n,

$p_t^1 \geq 0$, t=1, 2, …, T. (22)

2.4.2. Aggressive strategy

The investor aiming for an aggressive strategy is more prone to risk, being more concerned with return than risk.

Mean-absolute deviation model (with a pessimistic stance regarding risk)

Max ($\beta\left(\sum_{i=1}^{n} r_i^U x_i\right)$ - α($\lambda \frac{1}{T}\sum_{t=1}^{T} v_t + (1-\lambda)u^U$))

s.t. $\sum_{i=1}^{n} x_i$=1,

$\sum_{i=1}^{n} y_i \leq h^U$,

$x_i \leq u_i^U y_i$, i = 1, …, n,

$\sum_{i=1}^{n} r_{it}\, x_i + e_t^{L-} - e_t^{L+}$=$\sum_{i=1}^{n} r_i^L x_i$, t=1, ..., T,

$\sum_{i=1}^{n} r_{it}\, x_i + e_t^{U-} - e_t^{U+}$=$\sum_{i=1}^{n} r_i^U x_i$, t=1, ..., T,

$e_t^{U-} + e_t^{U+} \leq v_t$, t=1, 2, …, T,

$e_t^{L-} + e_t^{L+} \leq v_t$, t=1, 2, …, T,

$v_t \leq u^U$,

$x_i \geq 0$, i = 1, …, n,

$y_i \in \{0,1\}$, i = 1, …, n,

$0 \leq \lambda \leq 1$,

$v_t \geq 0$ t = 1, 2, … , T. (23)

Mean-semi-absolute deviation model (with an optimistic stance regarding risk)

Max ($\varphi\sum_{i=1}^{n} r_i^U x_i - \tau\, \frac{1}{T}\sum_{t=1}^{T} p_t^1$), s.t. $\sum_{i=1}^{n} x_i$=1,

$\sum_{i=1}^{n} y_i \leq h^U$,

$x_i \leq u_i^U y_i$, i = 1, …, n,

$p_t^1 + \sum_{i=1}^{n}(r_{it} - r_i^L)\, x_i \geq 0$, t=1, 2, …, T,

$x_i \geq 0$, i = 1, …, n,

$y_i \in \{0,1\}$, i = 1, …, n,

$p_t^1 \geq 0$, t=1, 2, …, T. (24)

2.4.3. Combined strategy

A combined strategy allows for the investor to choose a more balanced approach regarding risk and return.

Mean-absolute deviation model (with a pessimistic stance regarding risk)

Max ρ $(\beta(\sum_{i=1}^{n} r_i^L\, x_i))$+(1-ρ) $\beta\left(\sum_{i=1}^{n} r_i^U\, x_i\right) - \alpha(\lambda \frac{1}{T}\sum_{t=1}^{T} v_t + (1-\lambda)u^U)$

s.t. $\sum_{i=1}^{n} x_i$=1,

$\sum_{i=1}^{n} y_i \leq h^U - \delta(h^U - h^L)$,

$x_i + (-u_i^U + \delta_i(-u_i^L + u_i^U))y_i \leq 0$, i = 1, …, n,

$\sum_{i=1}^{n} r_{it}\, x_i + e_t^{L-} - e_t^{L+}$=$\sum_{i=1}^{n} r_i^L x_i$, t=1, ..., T,

$\sum_{i=1}^{n} r_{it}\, x_i + e_t^{U-} - e_t^{U+}$=$\sum_{i=1}^{n} r_i^U x_i$, t=1, ..., T,

$e_t^{U-} + e_t^{U+} \leq v_t$, t=1, 2, …, T,

$e_t^{L-} + e_t^{L+} \leq v_t$, t=1, 2, …, T,

$v_t \leq u^U$,

$x_i \geq 0$, i = 1, …, n,

$y_i \in \{0,1\}$, i = 1, …, n,

$0 \leq \lambda \leq 1$,

$v_t \geq 0,\ \mathrm{t} = 1, 2, \ldots, \mathrm{T}$, (25)

where ρ, δ and δ_i, i = 1, ...n, are indexes of pessimism ranging in a scale from 0 (agressive strategy) to 1 (conservative strategy).

Mean-semi-absolute deviation model

$$\text{Max } \rho\,(\varphi \sum_{i=1}^{n} r_i^L x_i - \tau \frac{1}{T}\sum_{t=1}^{T} p_t^2) + (1-\rho)\,(\varphi \sum_{i=1}^{n} r_i^U x_i - \tau \frac{1}{T}\sum_{t=1}^{T} p_t^1)$$

$$\text{s.t. } \sum_{i=1}^{n} x_i = 1,$$

$$\sum_{i=1}^{n} y_i \le h^U - \delta(h^U - h^L),$$

$$x_i + (-u_i^U + \delta_i(-u_i^L + u_i^U))y_i \le 0,\ i = 1, \ldots, n,$$

$$p_t^1 + \sum_{i=1}^{n}(r_{it} - r_i^L)\, x_i \ge 0,\ t=1, 2, \ldots, T,$$

$$p_2^1 + \sum_{i=1}^{n}(r_{it} - r_i^U)\, x_i \ge 0,\ t=1, 2, \ldots, T,$$

$$x_i \ge 0,\ i = 1, \ldots, n,$$

$$y_i \in \{0,1\},\ i = 1, \ldots, n,$$

$$p_t^1 \ge 0,\ t=1, 2, \ldots, T,$$

$$p_t^2 \ge 0,\ t=1, 2, \ldots, T. \qquad (26)$$

3. Data and Assumptions

A sample of ten complete years (about 3600 trading days) regarding a portfolio of diversified stocks from different countries and distinct activity sectors has been considered, using Bloomberg as a source of information. The stocks contemplated for the European countries are Portugal Telecom SGPS SA (PT)[1], Energias de Portugal, SA (EDP), Deutsche Telekom AG (DT), Bayerische Motoren Werke AG (BMW) and Adidas AG. For the United States of America we have collected data for Microsoft Corporation, Apple Incorporation, McDonald's Corporation, The Coca-cola company and Pfizer Incorporation. Regarding the Emergent markets, we have assessed Petrobras, Vale SA, America Movil SAB, Sify Technologies Limited and Taiwan semiconductor manufacturing company.

The selection of stocks for the portfolio analyzed has been done by considering several levels of diversification. Therefore, stocks from 15 companies have been carefully chosen from distinct geographical locations (7 countries and 4 continents), facing different "development stages" (10 developed and 5 emerging countries), and belonging to differentiated activity sectors and industries (see Tables 1 and 2).

Table 1: Industry classification of the stocks selected.

Industry Classification	**Companies**

[1] On May 29, 2015, the shareholders decided to change the name to PHAROL, SGPS S.A.

Communication Services	PT
Utilities - Regulated	EDP
Communication Services	DT
Auto Manufacturers - Major	BMW
Textile-Apparel Footwear & accessories	Adidas
Application Software	Microsoft
Computer Hardware	Apple
Restaurants	McDonalds
Beverages - Non-Alcoholic	Coca-Cola
Drug Manufacturers	Pfizer
Oil & Gas - Integrated	Petrobras
Metals & Mining	Vale
Communication Services	America
Communication Services	Sify
Semiconductors	Taiwan

Table 2: Covariance Matrix.

Covariance Matrix															
Companies	**PT**	**EDP**	**DT**	**BMW**	**Adidas**	**Microsoft**	**Apple**	**McDonalds**	**Coca-Cola**	**Pfizer**	**Petrobras**	**Vale**	**America**	**Sify**	**Taiwan**
PT	1														
EDP	0,355328	1													
DT	0,399601	0,290327	1												
BMW	0,329878	0,313495	0,436421	1											
Adidas	0,262390	0,275277	0,342802	0,500040	1										
Microsoft	0,157448	0,184613	0,344188	0,340842	0,282786	1									
Apple	0,138446	0,109123	0,219309	0,242284	0,209032	0,479262	1								
McDonalds	0,155646	0,143568	0,197158	0,285755	0,213723	0,351208	0,272031	1							
Coca-Cola	0,150945	0,146243	0,226163	0,244648	0,182934	0,421602	0,286638	0,341024	1						
Pfizer	0,169203	0,169098	0,261595	0,287335	0,229526	0,421951	0,271174	0,342997	0,404562	1					
Petrobras	0,206171	0,233812	0,231939	0,309524	0,242749	0,285464	0,258415	0,213090	0,232772	0,265068	1				
Vale	0,190548	0,210904	0,182029	0,286247	0,260905	0,274313	0,286622	0,216261	0,226315	0,240606	0,606503	1			
America	0,188983	0,194312	0,261892	0,327771	0,258780	0,349734	0,320524	0,271020	0,293091	0,291124	0,418344	0,418062	1		
Sify	0,127993	0,146724	0,183956	0,199308	0,173747	0,212633	0,210300	0,127042	0,130316	0,147408	0,210081	0,198388	0,215855	1	
Taiwan	0,130818	0,154492	0,063762	0,128628	0,144204	0,030031	0,047132	0,032455	-0,019781	0,024966	0,106917	0,108258	0,106691	0,153537	1

The assets' returns have been obtained through the computation of the daily logarithmic returns:

$$r_{it} = \ln(p_t) - \ln(p_{t-1}) = ln\frac{p_t}{p_{t-1}}. \tag{27}$$

Out of the various factors influencing the rate of change in expected return of the assets, the periods under which the returns are obtained might have a huge impact on the portfolio return. Therefore, returns obtained during the periods of economic crises and during periods of economic recuperation should be considered separately, since many investors may plan their asset allocation considering the return of the assets according to bull and bear scenarios. The usual delimitation of the periods of crises is based on remarkable events, as the Lehman Brothers collapse, for instance. Specifically, we consider the period between an early stage of the pre-crisis boom until a late stage of the boom, i.e. from 2001 (the starting year of our data) until 8th of August 2007 (the eve of the Lehman Brothers collapse), the crisis period as the time horizon between 9th of August 2007 and 31th of December 2009 and from then on until the latest years of our data (4th of April 2013) a "recovery" stage (Milesi-Ferretti and Tille, 2011; Mobarek et al., 2014).

The average returns obtained for the economic periods of pre-crisis, crisis and post crises are, respectively: $r_{79}=\sum_{t=1}^{79}\frac{1}{79}r_{it}x_i$, $r_{28}=\sum_{t=80}^{108}\frac{1}{28}r_{it}x_i$ and $r_{27}=\sum_{t=109}^{136}\frac{1}{27}r_{it}x_i$.

From these values it was possible to obtain the upper and lower bounds considered for the interval objective function coefficients for the return to be maximized, i.e. $r_i^L=\min\{r_{79},r_{28},r_{27}\}$ and $r_i^U=\max\{r_{79},r_{28},r_{27}\}$ (see Figure 1).

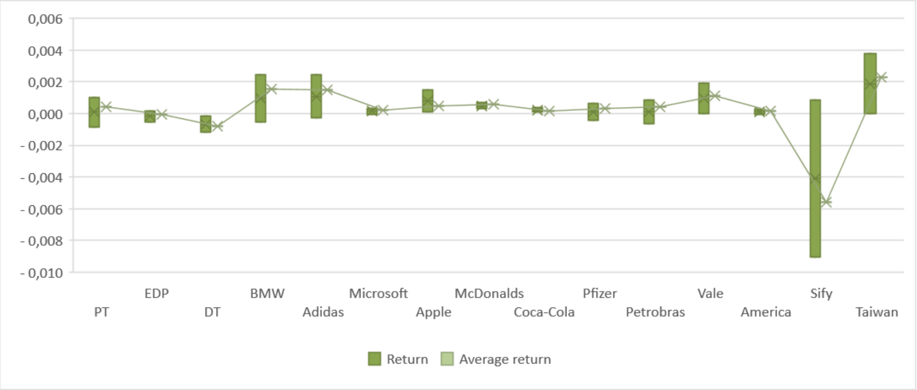

Figure 1. Ranges of variation of log return and log average return of the stocks selected.

The number of assets that the investor wants to manage in a portfolio is given within $[h^L, h^U]=[5,6]$ (Gupta et al., 2014). Finally, the maximum proportion of capital allocated to the assets in the portfolio is considered within the interval $[u_i^L,u_i^U]=[20\%,50\%]$, in order to obtain a certain level of investments' diversification.

4. Results and discussion

Table 3 provides information regarding the efficient Portfolios selected according to distinct risk profiles (other scenarios could be explored) considering the semi-mean-absolute deviation model.

Table 3: Proportion of assets in the obtained portfolios using different strategies with semi-mean-absolute deviation model.

	Portfolio 1	Portfolio 2	Portfolio 3	Portfolio 4	Portfolio 5
Strategy	**Conservative**	**Aggressive**	**Conservative**	**Aggressive**	**Combined**
Companies	**φ=1, τ=0**	**φ=1, τ=0**	**φ=0, τ=1**	**φ=0, τ=1**	**All indexes = 1/2**
PT	0.000	0.000	0.000	0.000	0.00
EDP	0.000	0.000	0.200	0.288	0.00
DT	0.000	0.000	0.000	0.000	0.00
BMW	0.000	0.500	0.000	0.000	0.00
Adidas	0.000	0.000	0.000	0.097	0.00
Microsoft	0.000	0.000	0.200	0.000	0.00
Apple	0.200	0.000	0.000	0.000	0.15
McDonalds	0.200	0.000	0.200	0.000	0.32
Coca-Cola	0.200	0.000	0.200	0.262	0.00
Pfizer	0.000	0.000	0.000	0.149	0.00
Petrobras	0.000	0.000	0.000	0.000	0.00
Vale	0.200	0.000	0.000	0.157	0.28
America	0.000	0.000	0.200	0.000	0.00
Sify	0.000	0.000	0.000	0.047	0.00
Taiwan	0.200	0.500	0.000	0.000	0.25

Portfolios 1 and 3 are obtained considering a more conservative strategy attaining the lower bound of maximum return and the upper bound of minimum risk, respectively, in a worst case scenario; while Portfolios 2 and 4 are more aggressive since they seek to compute the upper bound of the maximum return and the lower bound of the minimum risk in a best case scenario, respectively.

Regarding Portfolio 1 it is possible to conclude that the proportions of the selected assets are distributed evenly among a diversified set of stocks belonging to different industries (see Table 1). This fact is highlighted by the covariance between Taiwan/Coca-Cola (-0.02) which has negative values, followed by near zero covariance values between Taiwan and the remaining companies, in particular: Microsoft (0.02); McDonalds (0.03) and Vale (0.11) – see Table 2.

An aggressive strategy aiming for return only, leads to the choice of two blue-chip stocks, but with lower diversification level (see Portfolio 2).

A regulated utility is always selected when the investor is strictly seeking risk minimization (Portfolios 3 and 4). From the Portfolios analyzed the highest diversification is obtained when a more aggressive strategy towards risk is assumed (see Portfolio 4). While Portfolio 3 leads to a uniform distribution of the proportion of assets held, Portfolio 4 assigns a higher weight to EDP (regulated utility) and Coca-cola (which present a beta substantially lower than one).

The combined strategy considers a balanced approach by assigning a pessimistic index of 0.5 to all the objective functions and to the threshold of the interval constraints. The portfolio selected corresponds to the highest proportion of investment allocated to stocks which are also present in the portfolios obtained with the conservative strategy.

The expected trade-off between risk and return are highlighted in Figure 2, illustrating that portfolios with higher risk also obtain higher return. These results are also consistent with the type of strategy followed by the investor with more aggressive strategies toward risk leading to lower risk and more aggressive strategies toward return leading to higher return and vice-versa.

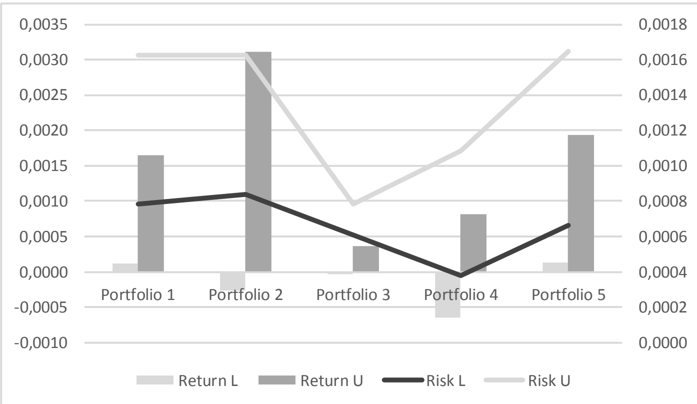

Figure 2. Risk vs. Return in the semi-mean-absolute model.

Table 4 presents information regarding the efficient Portfolios selected according to distinct risk profiles (once more, other scenarios could also be explored) considering the absolute deviation model.

Table 4: Proportion of assets in the obtained portfolios using different strategies with mean-absolute deviation model.

	Portfolio 6	**Portfolio 7**	**Portfolio 8**	**Portfolio 9**	**Portfolio 10**
Strategy	**Conservative**	**Aggressive**	**Conservative**	**Aggressive**	**Combined**
Companies	φ=1, τ=0	φ=1, τ=0	φ=0, τ=1	φ=0, τ=1	**All indexes = 1/2**

PT	0,000	0,000	0,000	0,000	0,00
EDP	0,000	0,000	0,200	0,164	0,00
DT	0,000	0,000	0,000	0,120	0,00
BMW	0,000	0,500	0,000	0,000	0,00
Adidas	0,000	0,000	0,000	0,000	0,00
Microsoft	0,000	0,000	0,200	0,038	0,00
Apple	0,200	0,000	0,000	0,000	0,07
McDonalds	0,200	0,000	0,200	0,100	0,35
Coca-Cola	0,200	0,000	0,200	0,500	0,35
Pfizer	0,000	0,000	0,000	0,000	0,00
Petrobras	0,000	0,000	0,000	0,000	0,00
Vale	0,200	0,000	0,000	0,000	0,05
America	0,000	0,000	0,200	0,078	0,18
Sify	0,000	0,000	0,000	0,000	0,00
Taiwan	0,200	0,500	0,000	0,000	0,00

Analogously to the previous model formulation, Portfolios 6 and 8 are obtained considering a more conservative strategy attaining the lower bound of maximum return and the upper bound of minimum risk, respectively, under a worst case scenario; while Portfolios 7 and 9 are more aggressive since they seek to compute the upper bound of the maximum return and the lower bound of the minimum risk under a best case scenario, respectively.

From the analysis of Table 3 it can be concluded that Portfolios 6-8 are identical to the ones obtained with the previous model. When the maximization of return takes place the results are always similar even with less stringent constraints regarding the portfolio diversification. However, if risk minimization is considered the results are only similar under a worst case scenario (because the feasible region becomes too tight). Finally, it can be established that with this last model formulation the values of the lower bound of return (i.e. in a worst case scenario) are always higher even with a combined strategy (see Figure 3).

Overall, results show that by following a conservative strategy the Portfolios selected are always diversified, leading to the management of 5 stocks (the upper limit considered in this scenario).

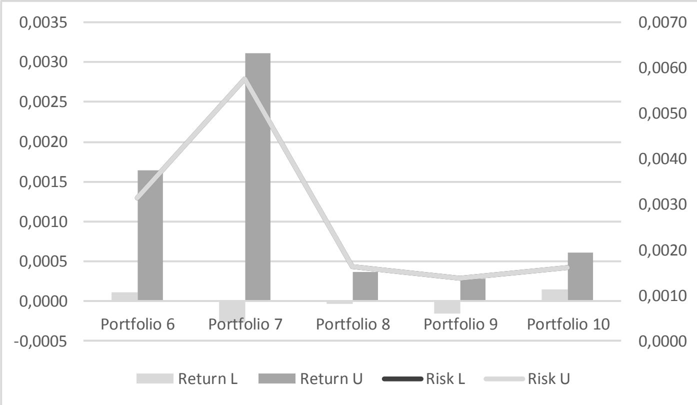

Figure 3. Risk vs. Return in the mean-absolute model.

Both model formulations have also been used by considering the average log return of the stocks assessed. The results obtained regarding the Portfolio composition are provided in Tables 5 and 6.

Table 5: Proportion of assets in the obtained portfolios using different strategies with semi-mean-absolute deviation model.

	Portfolio 11	**Portfolio 12**	**Portfolio 13**	**Portfolio 14**	**Portfolio 15**
Strategy	**Conservative**	**Aggressive**	**Conservative**	**Aggressive**	**Combined**
Companies	φ=1, τ=0	φ=1, τ=0	φ=0, τ=1	φ=0, τ=1	**All indexes = 1/2**
PT	0,000	0,000	0,000	0,000	0,00
EDP	0,000	0,000	0,200	0,223	0,00
DT	0,000	0,000	0,200	0,130	0,00
BMW	0,200	0,500	0,000	0,000	0,00
Adidas	0,200	0,000	0,000	0,000	0,13
Microsoft	0,000	0,000	0,000	0,000	0,00
Apple	0,000	0,000	0,000	0,000	0,10
McDonalds	0,200	0,000	0,000	0,000	0,18
Coca-Cola	0,000	0,000	0,200	0,500	0,00
Pfizer	0,000	0,000	0,200	0,000	0,00
Petrobras	0,000	0,000	0,000	0,000	0,00
Vale	0,200	0,000	0,200	0,071	0,25
America	0,000	0,000	0,000	0,075	0,00
Sify	0,000	0,000	0,000	0,001	0,00
Taiwan	0,200	0,500	0,000	0,000	0,35

Table 6: Proportion of assets in the obtained portfolios using different strategies with mean-absolute deviation model.

Portfolio 16	**Portfolio 17**	**Portfolio 18**	**Portfolio 19**	**Portfolio 20**

Strategy	Conservative	Aggressive	Conservative	Aggressive	Combined
Companies	**φ=1, τ=0**	**φ=1, τ=0**	**φ=0, τ=1**	**φ=0, τ=1**	**All indexes = 1/2**
PT	0,000	0,000	0,000	0,000	0,00
EDP	0,000	0,000	0,200	0,221	0,00
DT	0,000	0,000	0,200	0,130	0,00
BMW	0,200	0,500	0,000	0,000	0,00
Adidas	0,200	0,000	0,000	0,000	0,00
Microsoft	0,000	0,000	0,000	0,000	0,00
Apple	0,000	0,000	0,000	0,000	0,06
McDonalds	0,200	0,000	0,000	0,000	0,35
Coca-Cola	0,000	0,000	0,200	0,500	0,30
Pfizer	0,000	0,000	0,200	0,000	0,00
Petrobras	0,000	0,000	0,000	0,000	0,00
Vale	0,200	0,000	0,200	0,069	0,18
America	0,000	0,000	0,000	0,080	0,12
Sify	0,000	0,000	0,000	0,000	0,00
Taiwan	0,200	0,500	0,000	0,000	0,00

Since the objective functions are no longer interval valued, a more conservative strategy corresponds to the adoption of the model with the most constrained version of the feasible region (see Portfolios 11, 13, 16 and 18), i.e., using the worst case scenario; while more aggressive portfolios are obtained with the widest version of the feasible region, i.e. using a best case scenario (see Portfolios 12, 14, 17 and 19).

For the same reasons previously mentioned, Portfolios 11-13 are identical to Portfolios 16-18. However, it was possible to conclude that once more, with the mean-absolute modelling formulation the lower bounds of the returns of the portfolios selected are higher than those attained with the mean-semi-absolute formulation (in particular, when we contrast Portfolios 14 and 15 with Portfolios 19 and 20), leading us to conclude that the investor less prone to risk, i.e. which aims to obtain the highest returns considering the worst case scenario should select the absolute modelling approach.

5. Conclusions

A new modelling framework based on portfolio theory was proposed which accounts for the uncertainty handling by means of interval coefficients both in the objective functions and in the constraints. A new solution method for obtaining efficient portfolios has also been suggested which allows considering three types of investment strategies, i.e. a conservative strategy, an aggressive strategy and a combined strategy. This study contributes to an understanding of the implications of investment decisions under different risk assumption (i.e. mean-absolute deviation and mean-semi-absolute deviation). Our results show that under a conservative strategy (i.e. with higher risk concerns) the portfolios obtained are always more diversified, leading to less extreme asset allocations. Furthermore, our findings highlight the trade-off between risk and return, indicating that depending on the strategy followed by the investor, higher risk also generates higher return and vice-versa. Future work is currently under way in order to encompass other axes of evaluation, such as liquidity, considering multi-period portfolio selection models which allow exploring the investor's preferences in distinct economic cycles.

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